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Iterative Algorithms for Attitude Estimation Using Global Positioning System Phase Measurements

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Algorithms are sought for attitude determination using global positioning system (GPS) differential phase measurements, assuming that the cycle integer ambiguities are known. The problem of attitude determination is posed as a constrained parameter optimization problem, where a quaternion-based quartic cost function is used. A new general minimization scheme is developed. The new scheme is a continuous version of the well-known Newton–Raphson algorithm and is based on the solution of an ordinary differential equation. The new continuous algorithm converges exponentially from any initial condition to the closest local minimum located on the gradient direction in regions where the associated Hessian matrix is positive definite. Three new algorithms are developed for solving the attitude estimation problem, a discrete Newton–Raphson-based algorithm, a continuous Newton–Raphson algorithm, and an algorithm that is based on the eigenproblem structure of the nonlinear equations, which are related to the minimization of the quartic cost function. The performance of the new algorithms is evaluated via numerical examples and compared with each other and against the well-known QUEST algorithm. The continuous Newton–Raphson algorithm and the eigenproblem algorithm have similar accuracy. The discrete Newton–Raphson algorithm is less efficient than the continuous Newton–Raphson algorithm in the examined minimization because its search may wander and may even reach a nonrelevant extreme. When the GPS satellites are at low elevation, the accuracy of the new algorithms is better than that of QUEST, when the latter is applied to vectorized phase measurements.

I. Introduction

ATTITUDE determination using global positioning system (GPS) carrier signals has been given a considerable attention in the last decade.[1,2] Much attention was given to concept, hardware, and algorithm development, as well as to testing. Algorithms for GPS attitude determination given differential phase measurements can be broken into integer resolution and attitude calculations. Several methods for integer resolution are presented in the literature (e.g., see Refs. 1 and 3). In this work we assume that the integer ambiguity is solved, and we are concerned only with attitude calculation. The problem of attitude determination can be expressed as the problem of minimizing the following cost function[4] with respect to the attitude matrix D_a^e:

$$\rho\left(D_a^e\right) = \sum_i^n p_i \sum_j^2 \left|B_{ji} - \boldsymbol{a}_j^T D_a^e \boldsymbol{s}_i\right|^2 \tag{1}$$

The cost function is defined for n satellites and a planar array of three antennas. The vector $\boldsymbol{s}_i$ is a unit vector in the direction of the observed ith GPS satellite. The system e is the reference (Earth) coordinate system in which $\boldsymbol{s}_i$ is resolved. The coordinate system a is the antenna (body) coordinate system. Assuming that at least two of the lines connecting one antenna to the other (the baselines) are orthogonal, then the antenna arrangement constitutes a Cartesian coordinate system. The vector $\boldsymbol{a}_j$ is a vector along the jth axis of system a. The attitude matrix D_a^e is the transformation matrix from e to a. The measurement B_{ji} is the processed phase measurement defined as the projection of $\boldsymbol{s}_i$, the unit vector to satellite i, on the body coordinate system axis j. The weight p_i is a normalized weight given to the measurement related to the ith satellite, where

$$\sum_{i=1}^n p_i = 1$$

[The notation $(.)^T$ is used to denote the transpose of (.).] It is assumed that the components of $\boldsymbol{s}_i$ in e are known. Note that we are considering a planar problem in which only two components of B are measured for each satellite. These components are the projections of processed phase measurements on the body coordinate axes.

As described in Ref. 4, in the general case, there are more than three antennas and the baselines are not necessarily orthogonal. This paper, however, treats, without loss of generality, the minimal case of three planar antennas and two orthogonal baselines.

It was shown in the past[4] that the phase measurements could easily be converted into vector measurements, and then a least-squares fit could be found using one of the available algorithms such as QUEST.[5] Another possible approach is based on a least-squares fit of the attitude quaternion to the basic GPS phase measurements.[4] This approach was motivated by the success attained in quaternion fitting to vector measurements, which was achieved using the q-method solution.[6] However, unlike the case of vector measurements, where the cost function reduces to a quadratic form of a symmetric matrix, in the case of phase measurements, the cost function is a sum of quartic forms. Therefore, the q-method solution is not applicable in this case. A possible solution to the problem of finding the optimal quaternion, which minimizes this cost function, is an iterative one. Indeed, such a solution was presented in the literature.[4] That solution used the gradient projection technique to develop a steepest descent search for the local minimum of the cost function. It was found that this iterative process converged slowly. Therefore, a faster converging algorithm was sought in the present work.

The next section of this paper follows Ref. 4 and briefly presents the conversion of the raw phase measurements into vector measurements in the case where the antennas physically define a Cartesian coordinate system. This conversion is necessary if the Wahba[7] formulation and the QUEST algorithm are used. For the sake of comparison between QUEST, which operates on measured vectors, and the algorithms developed in the paper, which operate on raw phase measurements, a short description of QUEST is also given. Section III briefly presents the derivation of the quaternion-based quartic cost function,[4] which is related to the raw phase

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measurements. Section IV describes the necessary conditions for minimizing the quartic cost function and the nonlinear equations associated with the constrained minimization of that function. Section V presents a Newton–Raphson approach to the solution of the minimization problem. This approach yields a new discrete iterative minimization algorithm, which is shown to fail in certain cases where the initial guess is far from the correct minimizing quaternion. In view of this possible disadvantage, another new scheme is introduced, which is a continuous version of the Newton–Raphson algorithm. The continuous algorithm is based on the solution of an ordinary differential equation (ODE) where its steady solution state is the desired local minimum. It is shown that the continuous algorithm converges exponentially from any initial condition to the closest local minimum located on the gradient direction, in regions where the associated Hessian matrix is positive definite. The section also points out that the discrete Newton–Raphson algorithm can be obtained from the continuous version by a special selection of the integration step when using Euler integration scheme. In Sec. VI, we exploit the eigenproblem structure of the nonlinear equations described in Sec. IV to derive a new special algorithm. The performance of the new algorithms is evaluated via numerical examples and presented in Sec. VII. Four algorithms are compared. They are the QUEST algorithm, which is applied to the vectorized measurements, the eigenprolem algorithm, the discrete Newton–Raphson algorithm, and the continuous Newton–Raphson algorithm. Conclusions from the work of the paper are finally given in Sec. VIII.

II. Attitude Determination Using GPS Vectorized Observations

Several efficient algorithms for attitude determination based on a least-squares fit of the attitude to vector measurements have been introduced in the past. To make use of these algorithms, the phase measurements have to be converted into vector measurements in the body coordinate system as follows[4]:

$$\boldsymbol{s}_{ia} = \begin{bmatrix} B_{1i} \\ B_{2i} \\ \left(1 - B_{1i}^2 - B_{2i}^2\right)^{\frac{1}{2}} \end{bmatrix} \tag{2}$$

Note that the third component of $\boldsymbol{s}_{ia}$ is chosen to be the positive root of the expression in parentheses. This is done because only the signals of those GPS satellites that are above the antenna plane, and thus in the positive direction of the $\boldsymbol{a}_3$ axis, are received by the antennas. The vector $\boldsymbol{s}_{ia}$, resolved in Earth reference coordinates, is denoted by $\boldsymbol{s}_{ie}$, which is equivalent to $\boldsymbol{s}_i$ in Eq. (1). The vector $\boldsymbol{s}_{ie}$ is easily computed because both the satellite and the vehicle positions are known in Earth coordinates. With the pairs $\boldsymbol{s}_{ia}$ and $\boldsymbol{s}_{ie}$ on hand, $i = 1, 2, \ldots, n$, one can replace Eq. (1) by the following cost function introduced by Wahba[7]:

$$\rho'\left(D_a^e\right) = \frac{1}{2}\sum_i^n p_i \left|\boldsymbol{s}_{ia} - D_a^e \boldsymbol{s}_{ie}\right|^2 \tag{3}$$

Using QUEST,[5] similar algorithms that use the q-method,[6] or other algorithms, one can obtain a weighted least-squares attitude quaternion fit, which minimizes ρ'. For the sake of comparison between QUEST, which operates on measured vectors and uses the q-method, and the algorithms of this paper, which operate on phase measurements, a brief description of q-method is given next.

Because D_a^e is a known function of the attitude quaternion[8] $\boldsymbol{q}$, then $\rho'(D_a^e)$ can be replaced by $w(\boldsymbol{q})$, where

$$w(\boldsymbol{q}) = \frac{1}{2}\sum_i^n p_i \left|\boldsymbol{s}_{ia} - D_a^e(\boldsymbol{q})\boldsymbol{s}_{ie}\right|^2 \tag{4}$$

It can be shown that $\boldsymbol{q}^*$, which minimizes $w(\boldsymbol{q})$, is the same $\boldsymbol{q}$ that maximizes the cost function

$$\eta(\boldsymbol{q}) = \boldsymbol{q}^T K \boldsymbol{q} \tag{5}$$

where

$$K = \left[\begin{array}{c|c} \Sigma - \sigma I_3 & \varsigma \\ \hline \varsigma^T & \sigma \end{array}\right] \tag{6}$$

$$\sigma = \sum_{i=1}^n p_i \boldsymbol{s}_{ia}^T \boldsymbol{s}_{ie} \tag{7a}$$

$$B = \sum_{i=1}^n p_i \boldsymbol{s}_{ia} \boldsymbol{s}_{ie}^T \tag{7b}$$

$$\Sigma = B + B^T \tag{7c}$$

$$\varsigma = \sum_{i=1}^n p_i (\boldsymbol{s}_{ia} \times \boldsymbol{s}_{ie}) \tag{7d}$$

The matrix I_n is the nth-order identity matrix. It turns out that $\boldsymbol{q}^*$ is the eigenvector that corresponds to the largest eigenvalue of K. QUEST is an algorithm that yields this $\boldsymbol{q}^*$.

Because a QUEST-type algorithm uses a conversion of the raw phase measurements to vector measurements, its accuracy degrades and depends on the elevation of the GPS satellites.[9] Using the phase measurements themselves can eliminate this dependence. This is, in fact, a justification for the use of the algorithms developed in the next sections of the paper.

III. Phase-Related Cost Function Conversion to a Quartic Form

Recall Eq. (1),

$$\rho\left(D_a^e\right) = \sum_i^n p_i \sum_j^2 \left|B_{ji} - \boldsymbol{a}_j^T D_a^e \boldsymbol{s}_i\right|^2 \tag{8}$$

where one wishes to find D_a^e that minimizes $\rho(D_a^e)$. Note that in contrast to Eq. (4), where the cost $w(\boldsymbol{q})$ was formulated as a function of vector measurements, here the cost function is based on the raw phase measurements. Because, as mentioned earlier, D_a^e is a known function of the attitude quarternion $\boldsymbol{q}$, then $\rho(D_a^e)$ can be replaced by $J(\boldsymbol{q})$, where

$$J(\boldsymbol{q}) = \sum_i^n p_i \sum_j^2 \left|B_{ji} - \boldsymbol{a}_j^T D_a^e(\boldsymbol{q})\boldsymbol{s}_i\right|^2 \tag{9}$$

To facilitate the search for the quarternion $\boldsymbol{q}^*$ that minimizes $J(\boldsymbol{q})$, the latter is now converted into an explicit function of $\boldsymbol{q}$. To meet this end, define

$$L_{ji} = \left[\begin{array}{c|c} E_{ji} - \mu_{ji} I_3 & \boldsymbol{p}_{ji} \\ \hline \boldsymbol{p}_{ji}^T & \mu_{ji} \end{array}\right] \tag{10}$$

where

$$C_{ji} = \boldsymbol{s}_i \boldsymbol{a}_j^T \tag{11a}$$

$$E_{ji} = C_{ji} + C_{ji}^T \tag{11b}$$

$$\boldsymbol{p}_{ji} = \boldsymbol{a}_j \times \boldsymbol{s}_i \tag{11c}$$

$$\mu_{ji} = \boldsymbol{a}_j^T \boldsymbol{s}_i \tag{11d}$$

It can be shown that[4]

$$\boldsymbol{a}_j^T D_a^e(\boldsymbol{q})\boldsymbol{s}_i = \boldsymbol{q}^T L_{ji} \boldsymbol{q} \tag{12}$$

Substitution of Eq. (12) into Eq. (9) yields

$$J(\boldsymbol{q}) = \sum_i^n p_i \sum_j^2 \left|B_{ji} - \boldsymbol{q}^T L_{ji} \boldsymbol{q}\right|^2 \tag{13}$$

Because B_{ji} is a scalar and $q^T q = 1$, one can write

$$\Phi_{ji} = B_{ji} I_4 \tag{14}$$

$$B_{ji} = q^T \Phi_{ji} q \tag{15}$$

Therefore,

$$B_{ji} - q^T L_{ji} q = q^T (\Phi_{ji} - L_{ji}) q \tag{16}$$

Using the definition

$$M_{ji} = p_i^{\frac{1}{2}} (\Phi_{ji} - L_{ji}) \tag{17}$$

where M_{ji} is a 4×4 symmetric matrix, Eq. (13) can be written in the more compact form

$$J(q) = \sum_i^n \sum_j^2 \left| q^T M_{ji} q \right|^2 \tag{18}$$

Equation (18) can also be written as

$$J(q) = q^T \left[\sum_i^n \sum_j^2 M_{ji} q q^T M_{ji} \right] q \tag{19}$$

Obviously, the problem of finding the matrix D_a^e that minimizes $\rho(D_a^e)$, defined in Eq. (1), has been transformed into finding q that minimizes $J(q)$ of either Eq. (18) or Eq. (19). Unfortunately, $J(q)$ is quartic in q, whereas the cost function, which has to be optimized when solving Wahba's problem,[5] is only quadratic in q. For this reason the q-method solution is not suitable in the present case. One needs to use some other methods for minimizing $J(q)$. A minimizing iterative algorithm was suggested in Ref. 4. This algorithm, which was based on the gradient projection technique,[10] converged quite slowly, and therefore, faster algorithms were sought.

IV. Conditions for the Minimization of the Quartic Cost Function

Consider the constrained optimization problem

$$\min_x f(x) \tag{20}$$

subject to

$$h(x) = 0 \tag{21}$$

where

$$x \in R^n, \qquad f(x) : R^n \to R, \qquad f(x) \in C^2$$

$$h(x) : R^n \to R^m, \qquad \{h_k(x)\}_{k=1}^m \in C^2, \qquad m < n$$

and $h_i(x)$ is the ith component of the vector $h(x)$.

Let x^* be a local minimum of $f(x)$ satisfying $h(x^*) = 0$. Assuming that the $n \times m$ matrix $h_x(x^*)$ defined as

$$h_x \triangleq \left[\frac{\partial h_1}{\partial x}, \ldots, \frac{\partial h_m}{\partial x} \right] \tag{22}$$

has rank m, then necessary conditions for x^* to be a local minimum are as follows.[11,12]

Condition 1:

$$L_x(x^*, \lambda^*) = 0 \tag{23}$$

$$L_\lambda(x^*, \lambda^*) = 0 \tag{24}$$

where

$$L(x, \lambda) = f(x) + \lambda^T h(x) \tag{25}$$

and $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_m]^T$ is a nonzero vector of real numbers called Lagrange multipliers. The derivatives and L_x and L_λ are defined by

$$L_x \triangleq \left[\frac{\partial L}{\partial x} \right] = f_x + h_x \cdot \lambda \tag{26}$$

$$L_\lambda \triangleq \left[\frac{\partial L}{\partial \lambda} \right] = h \tag{27}$$

where the $n \times 1$ matrix f_x is defined as

$$f_x \triangleq \left[\frac{\partial f}{\partial x} \right] \tag{28}$$

Condition 2:

$$y^T L_{xx}(x^*, \lambda^*) y \geq 0 \qquad \text{for} \qquad \left\{ y \middle| h_x^T(x^*) y = 0 \right\} \tag{29}$$

where

$$L_{xx} \triangleq \begin{bmatrix} \left(\frac{\partial^2 f}{\partial x_1^2} + \lambda^T \frac{\partial^2 h}{\partial x_1^2} \right) \cdots \left(\frac{\partial^2 f}{\partial x_n \partial x_1} + \lambda^T \frac{\partial^2 h}{\partial x_n \partial x_1} \right) \\ \vdots \\ \left(\frac{\partial^2 f}{\partial x_1 \partial x_n} + \lambda^T \frac{\partial^2 h}{\partial x_1 \partial x_n} \right) \cdots \left(\frac{\partial^2 f}{\partial x_n^2} + \lambda^T \frac{\partial^2 h}{\partial x_n^2} \right) \end{bmatrix} \tag{30}$$

Condition 2 is equivalent to the following condition[13]:

$$Y^T L_{xx}(x^*, \lambda^*) Y \geq 0 \tag{31}$$

where Y is defined by

$$S \triangleq Y Y^T \tag{32}$$

and

$$S = I_n - h_x(x^*) \left[h_x^T(x^*) h_x(x^*) \right]^{-1} h_x^T(x^*) \tag{33}$$

The matrix S is a projection matrix onto the null space of $h_x^T(x^*)$ whose rank is $n - m$. Observe that $Y \in R^{n \times (n-m)}$, which is defined by the spectral decomposition of S, can be used as an orthonormal basis for this null space of $h_x^T(x^*)$.

The solution of the optimization problem is equivalent to the solution of Eqs. (23) and (24) of condition 1 while condition 2 is satisfied.

Our optimization problem, as described in Sec. III, can be formulated as

$$\min_q J(q) \tag{34}$$

subject to

$$q^T q = 1 \tag{35}$$

where

$$J(q) = q^T C(q) q \tag{36}$$

$$C(q) = \sum_i^n \sum_j^2 M_{ji} q q^T M_{ji} \tag{37}$$

and $q^T q = 1$ is the normality constraint satisfied by the quaternion q.

The Lagrangian associated with this problem is defined as

$$L(q, \lambda) = J(q) + \lambda h(q) \tag{38}$$

where

$$h(q) = 2(1 - q^T q) \tag{39}$$

Thus,

$$L_q = J_q + h_q\lambda \tag{40}$$

$$L_\lambda = h(\boldsymbol{q}) \tag{41}$$

$$J_q = 4C(\boldsymbol{q})\boldsymbol{q} \tag{42}$$

$$h_q = -4\boldsymbol{q} \tag{43}$$

$$L_{qq} = 4[2C(\boldsymbol{q}) + D(\boldsymbol{q}) - \lambda I_4] \tag{44}$$

where

$$D(\boldsymbol{q}) = \sum_i^n \sum_j^2 \boldsymbol{q}^T M_{ji}\boldsymbol{q} M_{ji} \tag{45}$$

The terms involved in h_q can be used to derive an explicit expression for the matrix S, which is needed for the testing of condition 2:

$$S = I_4 - \boldsymbol{q}\boldsymbol{q}^T \tag{46}$$

The equations $L_q = 0$ and $L_\lambda = 0$ can be written explicitly as[9]

$$\left.\begin{aligned} C(\boldsymbol{q})\boldsymbol{q} &= \lambda\boldsymbol{q} \\ \boldsymbol{q}^T\boldsymbol{q} &= 1 \end{aligned}\right\} \boldsymbol{q} \in R^4, \qquad \lambda \in R \tag{47}$$

Observe that $\lambda = \boldsymbol{q}^T C(\boldsymbol{q})\boldsymbol{q} = J(\boldsymbol{q}) \geq 0$ and, therefore, $\min_q J(\boldsymbol{q}) = \lambda_{\min} \geq 0$. This observation is used later to initialize the minimization algorithms near the minimum point of the quartic cost function by selecting $\lambda_0 = 0$. Also observe that $C(\boldsymbol{q})$ should be nonnegative definite to guarantee $\lambda \geq 0$.

V. Minimization via the Newton–Raphson Algorithm

A. Minimization via the Discrete Newton–Raphson Algorithm[11,12]

Following the discussion of the preceding section, the solution of the optimization problem is equivalent to the solution of Eqs. (23) and (24), which are related to condition 1, while condition 2 is satisfied. A natural approach is to solve the equations of condition 1 and then to test condition 2.

Define

$$z \triangleq [\boldsymbol{q}^T \quad \boldsymbol{\lambda}^T]^T \tag{48}$$

$$L_z = \begin{bmatrix} L_q \\ L_\lambda \end{bmatrix} = \begin{bmatrix} f_q + \boldsymbol{h}_q \\ h(\boldsymbol{q}) \end{bmatrix} \tag{49}$$

$$L_{zz} = \begin{bmatrix} L_{qq} & L_{q\lambda} \\ L_{\lambda q} & L_{\lambda\lambda} \end{bmatrix} = \begin{bmatrix} L_{qq} & \boldsymbol{h}_q \\ \boldsymbol{h}_q^T & 0 \end{bmatrix} \tag{50}$$

Applying the discrete Newton–Raphson algorithm to solve Eqs. (23) and (24), one can write the following difference equation:

$$z_{k+1} = z_k - L_{zz}^{-1}(z_k)L_z(z_k) \tag{51}$$

Observe that $L_{zz} > 0$ guarantees the existence of L_{zz}^{-1} and satisfies condition 2 of Sec. IV. This can easily be proved as follows. The inequality $L_{zz} > 0$ implies that

$$\begin{bmatrix} \boldsymbol{y}_1^T & \boldsymbol{y}_2^T \end{bmatrix} \begin{bmatrix} L_{qq} & h_q \\ h_q^T & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \end{bmatrix} = \boldsymbol{y}_1^T L_{qq}\boldsymbol{y}_1 + 2\boldsymbol{y}_2^T h_q^T \boldsymbol{y}_1 > 0$$

$$\text{for all} \quad \boldsymbol{y}_1 \quad \text{and} \quad \boldsymbol{y}_2$$

If $\boldsymbol{y}_1$ belong to $\{\boldsymbol{y}_1 | h_q^T\boldsymbol{y}_1 = 0\}$, then $L_{zz} > 0$ is equivalent to $\boldsymbol{y}_1^T L_{qq}\boldsymbol{y}_1 > 0$. The last inequality is exactly condition 2.

Application of Eq. (50) to our optimization problem leads to

$$L_z = \begin{bmatrix} L_q \\ L_\lambda \end{bmatrix} = \begin{bmatrix} J_q + h_q\lambda \\ h \end{bmatrix} = \begin{bmatrix} 4[C(\boldsymbol{q})\boldsymbol{q} - \boldsymbol{\lambda}\boldsymbol{q}] \\ 2(1 - \boldsymbol{q}^T\boldsymbol{q}) \end{bmatrix} \tag{52}$$

$$L_{zz} = \left[\begin{array}{c|c} L_{qq} & h_q \\ \hline h_q^T & 0 \end{array}\right] = \left[\begin{array}{c|c} 4[2C(\boldsymbol{q}) + D(\boldsymbol{q}) - \boldsymbol{\lambda} I_4] & -4\boldsymbol{q} \\ \hline -4\boldsymbol{q}^T & 0 \end{array}\right] \tag{53}$$

The existence of L_{zz}^{-1} is guaranteed in the case of a planar array of at least three antennas with at least two nonparallel baselines and with measurements from at least two satellites. The positive definiteness of L_{zz} is not guaranteed and can be achieved only at the vicinity of a minimum point.

Notice that near the minimum point a convergence of order two is achieved for a quadratic approximation of the function to be minimized.[12] In the case where the function is essentially different from that approximation, the iterative search may wander and even diverge.[11,12] Although in our case the existence of $L_{zz}^{-1}(z_k)$ is guaranteed, one cannot guarantee the prevention of a wandering search, which may lead to a convergence to an extremal point, which is far from the minimum closest to the initial point z_0. The example in the next section will demonstrate such a situation. It is possible, however, to devise Newton–Raphson-type algorithms, involving second-order derivatives that possess local quadratic convergence without the aforementioned disadvantages. Such an algorithm is derived next.

B. Minimization via the Continuous Newton–Raphson Algorithm

The new algorithm is designed to prevent the wandering search of the discrete Newton–Raphson algorithm and to guarantee continuous decrease toward the minimum when the algorithm is initialized at the vicinity of the minimum point.

The new algorithm is based on the gradient flow concept.[14] It converges exponentially from any initial condition to the closest local minimum located on the gradient direction in regions where the Hessian matrix L_{zz} is positive definite. The algorithm is, in fact, a continuous version of the Newton–Raphson scheme and based on a solution of an ODE whose steady state is the desired local minimum. It is shown later that the well-known discrete Newton–Raphson algorithm can be obtained from the continuous version by a special selection of the integration step using the Euler integration scheme.

Consider the problem of finding the solution z of the equation $L_z(z) = 0$.

Define

$$v \triangleq \tfrac{1}{2} L_z^T L_z > 0 \tag{54}$$

then

$$\dot{v} = L_z^T \dot{L}_z = L_z^T L_{zz}\dot{z} \tag{55}$$

Assuming that L_{zz}^{-1} exists, one can devise

$$\dot{z} = -\tfrac{1}{2}\eta L_{zz}^{-1} L_z \tag{56}$$

where η is a constant positive predetermined scalar. Such a selection of a differential equation can guarantee that v is a Lyapunov function and that Eq. (56) has a stable equilibrium point. This statement is proven next.

Substitution of Eq. (56) into Eq. (55) yields

$$\dot{v} = -\tfrac{1}{2}\eta L_z^T L_z = -\eta v < 0 \tag{57}$$

Therefore, v decreases exponentially with time, and as a result, $L_z \to 0$. Notice that $\dot{z} \to 0$ as $L_z \to 0$. Therefore, $z^* = [\boldsymbol{q}^{*T} \quad \boldsymbol{\lambda}^{*T}]^T$, which satisfies $L_z(z^*) = 0$, is the asymptotic solution of Eq. (56).

Remarks:

1) Here $\dot{v} = -\eta v$ guarantees an exponential decrease of v.

2) Here $\dot{v} = 0$ if and only if $L_z^T L_z = 0$, which implies $L_z = 0$.

3) Observe that $\dot{z} \to 0$ as $L_z \to 0$. Therefore, z^*, which satisfies $L_z(z^*) = 0$, is the asymptotic solution of the differential equation associated with z.

4) Let z define a smooth vector field on R^{m+n}. Assume that z^* is the associated equilibrium point. Let $\Omega \in R^{m+n}$ be a compact neighborhood of z^*. Then v is a Lyapunov function on Ω, and z^* is a stable equilibrium point.

5) Consider the Euler integration scheme for which $\dot{z}$ is approximated by

$$\dot{z} \cong (z_{k+1} - z_k)/\Delta t \tag{58}$$

Using Eq. (56), one gets

$$(z_{k+1} - z_k)/\Delta t = -\tfrac{1}{2}\eta L_{zz}^{-1}(z_k)L_z(z_k) \tag{59}$$

Therefore,

$$z_{k+1} = z_k - \tfrac{1}{2}\eta \Delta t L_{zz}^{-1}(z_k)L_z(z_k) \tag{60}$$

Observe that a selection of the integration step as $\Delta t = 2/\eta$ leads to the well-known iterative Newton–Raphson algorithm as described in Eq. (51).

6) The minimum of f coincides with that of L when $h(\boldsymbol{q}) = 0$; therefore, if $\dot{L} \leq 0$, the algorithm converges to the closest local minimum located on the gradient direction. As can easily be shown, this condition is satisfied if $L_{zz} > 0$. The time derivative of L is $\dot{L} = L_z^T \dot{z}$. Substitution of $\dot{z}$ from Eq. (56) leads to $\dot{L} = -\frac{1}{2}\eta L_z^T L_{zz}^T L_z$. Thus, $\dot{L} \leq 0$ if $L_{zz} > 0$. Recall that $L_{zz} > 0$ implies that condition 2 of Sec. IV, which is a necessary condition for a minimum, is satisfied.

VI. Eigenproblem Solution of the Nonlinear Equations

Equation (47) has a structure that calls for an iterative solution. One such solution that immediately comes to mind is the following. Guess an initial $\boldsymbol{q}$ and use it in Eq. (37) to compute $C(\boldsymbol{q})$. Then, find the eigenvalues and eigenvectors of $C(\boldsymbol{q})$. Reference 9 shows that J is minimal when $\boldsymbol{q}$ is the unit eigenvector that corresponds to the smallest eigenvalue of $C(\boldsymbol{q})$. Therefore, select this eigenvector and use it as $\boldsymbol{q}$ for the next iteration.

The eigenvalues associated with the equation $C(\boldsymbol{q})\boldsymbol{q} = \lambda \boldsymbol{q}$ should be real and nonnegative since $\lambda = \boldsymbol{q}^T C(\boldsymbol{q})\boldsymbol{q} = J(\boldsymbol{q}) \geq 0$. Therefore, a necessary condition for the convergence of the algorithm is that the matrix $C(\boldsymbol{q})$ be symmetric and nonnegative definite. Such a matrix has real nonnegative eigenvalues.

The matrix $C(\boldsymbol{q})$ is indeed symmetric because the matrices M_{ji} and $\boldsymbol{q}\boldsymbol{q}^T$ that compose $C(\boldsymbol{q})$ [see Eq. (37)] are symmetric. In addition the matrix $C(\boldsymbol{q})$ is positive definite because

$$\boldsymbol{x}^T C(\boldsymbol{q})\boldsymbol{x} = \boldsymbol{x}^T\left(\sum_i^n \sum_j^2 M_{ji}\boldsymbol{q}\boldsymbol{q}^T M_{ji}\right)\boldsymbol{x}$$

$$= \sum_i^n \sum_j^2 \left(\boldsymbol{x}^T M_{ji}\boldsymbol{q}\boldsymbol{q}^T M_{ji}\boldsymbol{x}\right) = \sum_i^n \sum_j^2 \left(\boldsymbol{x}^T M_{ji}\boldsymbol{q}\right)\left(\boldsymbol{x}^T M_{ji}\boldsymbol{q}\right)^T$$

$$= \sum_i^n \sum_j^2 \left(\boldsymbol{x}^T M_{ji}\boldsymbol{q}\right)^2 \geq 0 \qquad \text{for all} \quad \boldsymbol{x} \in R^4$$

It was found that the convergence of the preceding algorithm was very slow and characterized by oscillations. The oscillations, in terms of the angular error associated with the quaternion estimation error, were about the slowly converged average and started with amplitude of 5 deg that decreased slowly as the number of the iterations increased. Based on this observation, the algorithm was modified in the following way. The solutions obtained from two successive iterations were averaged. The average solution was then fed into the iterative algorithm, which was run twice again. The results of these two iterations were averaged again and so on.

A step-by-step description of the modified algorithm is as follows:

1) Guess $\boldsymbol{q}_k$, and set $k = 0$.

2) Compute $C_k(\boldsymbol{q}_k)$.

3) Find the eigenvalues and eigenvectors of $C_k(\boldsymbol{q}_k)$.

4) Set $\boldsymbol{q}_{k+1}$ to the eigenvector corresponding to the smallest eigenvalue of $C_k(\boldsymbol{q}_k)$.

5) Go once more to step 2, repeat steps 3 and 4.

6) Replace $\boldsymbol{q}_{k+1}$ by the average of the last two $\boldsymbol{q}_{k+1}$.

7) If $|\boldsymbol{q}_{k+1} - \boldsymbol{q}_k| \leq \delta$, where δ is a predetermined constant, then stop; otherwise increase the argument by 1 and go back to step 2.

The preceding new algorithm has no formal convergence proof, but it converged almost exponentially in all of the examined examples.

VII. Examples

A. Data Used in the Example

Given are five GPS satellites, which are used to generate the measurements. The vectors $\boldsymbol{s}_i$, $i = 1, 2, \ldots, 5$, are the unit vectors to these satellites. These vectors in the Earth reference coordinate system have the following values:

$$\boldsymbol{s}_1 = \begin{bmatrix} -0.2650 \\ 0.4589 \\ 0.8480 \end{bmatrix}, \qquad \boldsymbol{s}_2 = \begin{bmatrix} -0.7002 \\ 0.0984 \\ 0.7071 \end{bmatrix}, \qquad \boldsymbol{s}_3 = \begin{bmatrix} 0.5038 \\ 0.3027 \\ 0.8090 \end{bmatrix}$$

$$\boldsymbol{s}_4 = \begin{bmatrix} -0.3221 \\ -0.1571 \\ 0.9336 \end{bmatrix}, \qquad \boldsymbol{s}_5 = \begin{bmatrix} 0.3335 \\ -0.5337 \\ 0.7771 \end{bmatrix}$$

The vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are the x and y coordinate axes of the body system a, in which the computations are performed (see Sec. I). Thus,

$$\boldsymbol{a}_1^T = [1 \quad 0 \quad 0], \qquad \boldsymbol{a}_2^T = [0 \quad 1 \quad 0]$$

The transformation matrix D_a^e, from the reference to the body coordinates and its corresponding quaternion are

$$D_a^e = \begin{bmatrix} 0.7127 & 0.6588 & 0.2409 \\ -0.6797 & 0.5635 & 0.4696 \\ 0.1737 & -0.4984 & 0.8494 \end{bmatrix}$$

$$\boldsymbol{q}^T = [0.2738, -0.0190, 0.3786, 0.8840]$$

The corresponding Euler angles (degrees) of this attitude are

$$\psi = 42.7530, \qquad \theta = -13.9390, \qquad \phi = 28.9390$$

where the order of rotations is 3–2–1; that is, the sequence is the customary yaw, pitch, and roll angles.

For this satellite geometry the nominal phase measurements are

$$B_{11} = 0.3178, \qquad B_{12} = -0.2639, \qquad B_{13} = 0.7534$$

$$B_{14} = -0.1082, \qquad B_{15} = 0.0732$$

$$B_1 = 0.8369, \qquad B = 0.8634, \qquad B = 0.2081$$

$$B_{24} = 0.5688, \qquad B_{25} = -0.1624$$

The corrupted phase measurements are generated using zero mean Gaussian error. The standard deviations of the error associated with each of the five satellites are

$$\sigma_1 = 0.01, \qquad \sigma_2 = 0.05, \qquad \sigma_3 = 0.03$$

$$\sigma_4 = 0.02, \qquad \sigma_5 = 0.02$$

The errors themselves are errors in B_{ji}, the projection of $\boldsymbol{s}_i$, the unit vector to satellite i, on the body coordinate system axis j. We wish to find the quaternion that minimizes the cost function defined in Eq. (36). The weights p_i, associated with the cost function and used in Eqs. (1) and (17), are calculated via the relations

$$w_i = \frac{1}{\sigma_i^2}, \qquad W = \sum_{i=1}^5 w_i, \qquad p_i = \frac{w_i}{W} \quad \text{for} \quad i = 1, \ldots, 5$$

B. Behavior of the Quartic Cost Function

The cost function $J(\boldsymbol{q}) = \boldsymbol{q}^T C(\boldsymbol{q})\boldsymbol{q}$ of Eq. (36) is shown in Fig. 1, where the preceding data are used. Figure 1 presents $J(\boldsymbol{q})$ as a function of q_1 and q_2 in the region defined by $-1 \leq q_1 \leq 1$ and $-1 \leq q_2 \leq 1$, where q_3 is held constant and the normality constraint $\boldsymbol{q}^T\boldsymbol{q} = 1$ is satisfied. Because all of the relevant values of q_4 are defined by the normality constraint when q_1, q_2, and q_3 are given, q_4 is eliminated from Fig. 1. It was observed that the basic shape of the cost function, which contains a valley and a hill, is invariant to the value of q_3 as long the relation $\boldsymbol{q}^T\boldsymbol{q} = 1$ is satisfied. The particular

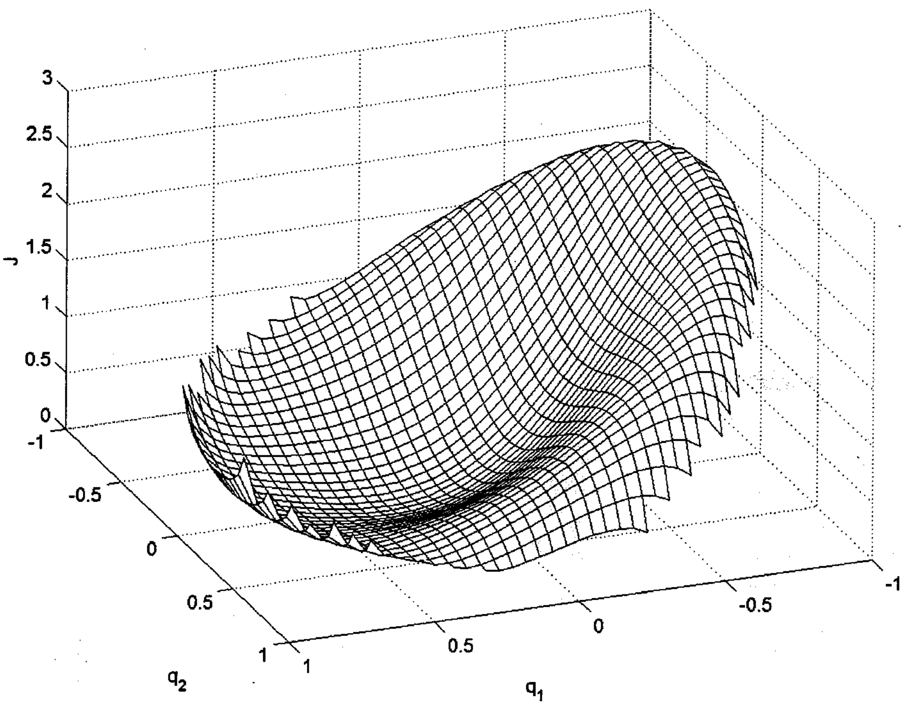

Fig. 1 Typical behavior of the cost function $J(q) = q^T C(q)q$.

value of q_3 used in Fig. 1 is 0.3809. As can be seen from Fig. 1, the cost function has two extreme points. One point is a minimum and the other one is a maximum.

C. Attitude Estimation Using Vectorized Phase Measurements

When the nominal phase measurements are vectorized according to Eq. (2) and QUEST is used to compute the optimal quaternion $\boldsymbol{q}^*$, the achieved accuracy is

$$|\boldsymbol{q} - \boldsymbol{q}^*| < 1 \times 10^{-15}$$

The same computation for the noisy phase measurements leads to the following quaternion error:

$$|\boldsymbol{q} - \boldsymbol{q}^*| < 0.0160, \qquad \phi_e = 1.8035$$

where ϕ_e (degrees) is the Euler angle associated with the quaternion error $\boldsymbol{q}_e = \boldsymbol{q}\tilde{\boldsymbol{q}}^*$ and $\tilde{\boldsymbol{q}}^*$ is the adjoint of $\boldsymbol{q}^*$.

The corresponding errors (in degrees) in yaw, pitch, and roll are

$$\delta\psi = -0.1526, \qquad \delta\theta = 0.3889, \qquad \delta\phi = -1.7453$$

D. Attitude Estimation Using Direct Phase Measurements

Instead of using the vectorized measurements and consequently the QUEST algorithm, here the attitude quaternion is estimated using the algorithms developed for finding $\boldsymbol{q}^*$ directly from the phase measurements themselves. These algorithms yield the quaternion that minimizes J of Eq. (36).

1. Attitude Estimation Using Newton–Raphson Algorithms

The characteristics of the cost function together with the structure of the antenna array and the number of the measured satellites influence the behavior of the Newton–Raphson algorithms for the following reasons.

1) The existence of $L_{zz}^{-1}(z)$ is guaranteed in the case of a planar array of at least three antennae with at least two nonparallel baselines and with measurements from at least two satellites.

2) The cost function is quartic and has one minimum and one maximum.

3) Here $\min_q J(\boldsymbol{q}) = \lambda_{\min} \geq 0$ and, therefore, $\lambda_0 = 0$ can be used to initialize the minimization algorithms near the minimum point.

Therefore, the continuous Newton–Raphson algorithm has a guaranteed exponential convergence to the single minimum of the cost function. Such a convergence is not guaranteed for the discrete Newton–Raphson algorithm because its search direction may jump and wander.

a. Attitude estimation using the discrete Newton–Raphson algorithm. The iteration starts with $\hat{\lambda}_0 = 0$ and a randomly chosen $\hat{\boldsymbol{q}}_0^T = [0.4266, 0.1606, 0.0454, 0.8889]$ where $\hat{\boldsymbol{q}}_0$ corresponds to the initial error $\phi_{e0} = 47.1081$ deg. This choice of the initial quaternion corresponds to the following initial angular errors (degrees):

$$\delta\psi = 29.7684, \qquad \delta\theta = -28.2260, \qquad \delta\phi = -23.9697$$

The solution of Eq. (51), where z is as given in Eq. (48), settles after 20 iterations on

$$\hat{\boldsymbol{q}}_{20}^T = [0.8391, 0.0674, -0.1815, -0.5083]$$

with angular error $\phi_{e20} = 146.3321$ deg and attitude estimation error (degrees)

$$\delta\psi = 24.9133, \qquad \delta\theta = -27.5905, \qquad \delta\phi = -35.6287$$

In this case, due to the wandering search that characterizes the discrete Newton–Raphson algorithm, the solution converges to the maximum rather to the minimum. A suitable change in the initialization point alters the behavior of the algorithm and results in a convergence to the minimum. This is demonstrated later in the section.

b. Attitude estimation using the continuous Newton–Raphson algorithm. The exact same data that have been used earlier is used here. Equation (56) is solved using $\eta = 10^8$, which sets the time constant of Eq. (57) to 10^{-8}, and a fourth-order Runge–Kutta integration scheme. It takes about 10 time constants for the algorithm to reach the steady-state value of the ODE, which is the minimum value of the cost function. The estimated attitude after 0.22 μs is

$$\hat{\boldsymbol{q}}^T(0.22\,\mu\text{s}) = [0.2815, -0.0167, 0.3809, 0.8806]$$

which yields a cost function value of 0.0003. The corresponding attitude estimation error (degrees) in terms of yaw, pitch, and roll is

$$\delta\psi = -0.2410, \qquad \delta\theta = 0.1735, \qquad \delta\phi = -0.9405$$

The quaternion error in this case is

$$|\boldsymbol{q} - \hat{\boldsymbol{q}}(0.22\,\mu\text{s})| = 0.0091, \qquad \phi_e(0.22\,\mu\text{s}) = 1.0404 \text{ deg}$$

2. Attitude Estimation Using the Iterative Eigenproblem Algorithm

When the iterations are started with the same initial conditions as before, the solution settles after five iterations on

$$\hat{q}_5^T = [0.2815, -0.0167, 0.3809, 0.8806]$$

which yields a cost function value of 0.0003. The corresponding final attitude estimation error in terms of yaw, pitch, and roll angles (degrees) is

$$\delta\psi = -0.2418, \qquad \delta\theta = 0.1736, \qquad \delta\phi = -0.9402$$

The quaternion error associated with this case is

$$|q - \hat{q}_5| = 0.0091, \qquad \phi_{e5} = 1.0403 \text{ deg}$$

The estimated attitude almost coincides with that of the continuous Newton–Raphson algorithm. The differences in the estimated Euler angles in both algorithms are of the order of 10^{-4} deg after 5 iterations and 0.22 μs and 10^{-7} deg after 10 iterations and 0.4 μs.

The new eigenproblem algorithm converged almost exponentially in all of the examined examples. The accuracy of both algorithms, the continuous Newton–Raphson algorithm and the eigenproblem algorithm, is better than that of QUEST due to the correlated measurement noise associated with the transformation to vectors.[15]

The convergence of the continuous Newton–Raphson algorithm vs that of the iterative eigenproblem and the discrete Newton–Raphson algorithms is presented in Fig. 2. In the examples presented in Fig. 2, we purposely chose a large initial error to demonstrate that there are cases where the discrete Newton–Raphson algorithm converges to the wrong extremum, whereas the continuous Newton–Raphson algorithm converges to the correct solution. For the sake

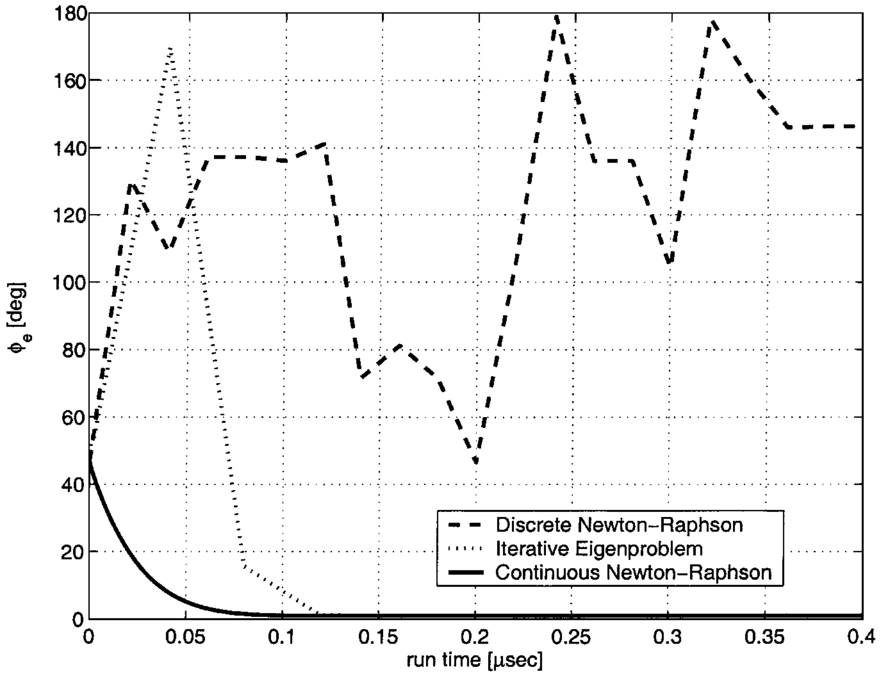

Fig. 2 Performance comparison of the iterative algorithms.

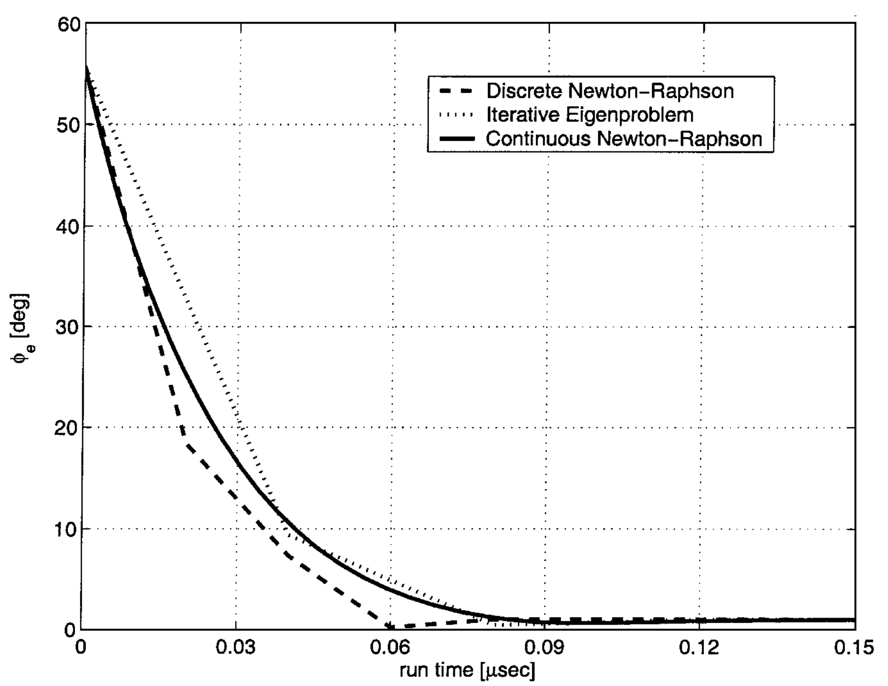

Fig. 3 Performance evaluation of the iterative algorithms using initial conditions different from those shown in Fig. 2.

of comparison the iterations of the discrete Newton–Raphson algorithm are converted to time using $\Delta t = 2/\eta$, where $\eta = 10^8$. To also compare the performance of the iterative eigenproblem algorithm with the other two algorithms, the time between the iterations is set to $2\Delta t$. The factor 2 is used because each iteration involves two eigenvalue/eigenvector calculations. Observe that η in Eq. (56) defines the stiffness. The integration method is selected according to the stiffness of the differential equation. It is possible to solve Eq. (56) for different values of η and use different integration schemes, which may simplify the implementation. The relative performance of each algorithm, as presented in Fig. 2, remains invariant. Figure 2 shows the performance of the various algorithms as a function of time where $\eta = 10^8$ and $\Delta t = 2/\eta = 0.02$ μs. The results presented in Fig. 2 as a function of run time can also be presented as a function of the iteration number. The conversion of run time to iterations in the continuous Newton–Raphson algorithm is done using the specific Δt value where the latter corresponds to one iteration. To find Δt, we use the relation $\Delta t = 2/\eta$, where in our case $\eta = 10^8$.

To demonstrate the case where also the discrete Newton–Raphson algorithm converges to the correct solution, we chose an initial quaternion that was closer to the correct one; namely, we chose $\hat{\boldsymbol{q}}_0 = [-0.0628, 0.2392, 0.1624, 0.9552]$ and $\hat{\lambda}_0 = 0$. The initial angular error that corresponds to this initial quaternion is $\phi_{e0} = 55.7138$ deg. Note that even though this initial angular error is larger than that of the earlier case, the geometry of the cost function J is such that from this initial point the discrete Newton–Raphson algorithm does converge to the correct quaternion. The convergence of the iterative algorithms in this case is shown in Fig. 3. All three algorithms converge to the minimum point with accuracy similar to that obtained before.

The solution of the equations associated with condition 1 of Sec. IV leads to

$$\hat{\boldsymbol{q}}^T = [0.2815, -0.0167, 0.3809, 0.8806]$$

This solution should also satisfy condition 2 of Sec. IV to guarantee a minimum. When Eqs. (46), (32), and (53) are used, it can be shown that

$$\lambda_i\left[Y^T L_{qq} Y\right] = \begin{bmatrix} 6.2178 \\ 1.2894 \\ 2.9601 \end{bmatrix}$$

Since $Y^T L_{qq} Y > 0$, then condition 2 is satisfied, and we have in this case a strict local minimum.

VIII. Conclusions

This paper deals with algorithms for attitude determination using GPS differential phase measurements, assuming that the cycle integer ambiguities are known. The problem of attitude determination is posed as a constrained parameter optimization problem where a suitable quaternion-based cost function is used. This cost function is based on a least-squares fit of the attitude quaternion to the GPS phase measurements themselves and has a quartic form. Unlike in the vectorized phase measurements case, Davenport's q-method is not applicable in this instance; therefore, new minimization algorithms are required. To meet this end, three new iterative minimization schemes were developed.

The first one is based on the well-known discrete Newton–Raphson algorithm.

The second algorithm is a continuous version of the Newton–Raphson algorithm. This algorithm is based on the solution of an ODE, where its steady state is the desired local minimum. The algorithm converges exponentially from any initial condition to the closest local minimum located on the gradient direction in regions where the associated Hessian matrix is positive definite. It was shown that the former discrete Newton–Raphson algorithm is a special case of the continuous version. The discrete algorithm is obtained from the continuous one by a special selection of the integration step when using the Euler integration scheme.

Finally, the third new algorithm presented in this work is one that is based on the eigenproblem structure of the nonlinear equations associated with the minimization of the quartic cost function.

The new algorithms were applied to the minimization of the quaternion-based cost function. Their performance was evaluated via numerical examples, and a comparison between them was made. It was found that the continuous Newton–Raphson algorithm and the eigenproblem algorithm had similar convergence rate and accuracy. Their accuracy was better than that of QUEST, particularly when the elevation of the GPS satellites was low. This is so not because of a deficiency in the q-method, or in QUEST, which implements this method, but rather because at low elevations the measurement errors are amplified by the conversion of the phase measurements to vector measurements. Such a conversion is needed when QUEST is used. Reference 15 proves that the transformation to vectors produces a correlated noise matrix, whereas Wahba's problem[7] assumes an isotropic measurement error covariance matrix.

It was shown that when the initial quaternion is far from the correct solution the discrete Newton–Raphson algorithm may converge to the wrong extremal point because the search direction may jump and miss the minimum.

The implementation of the continuous Newton–Raphson algorithm can be simplified by using an appropriate integration scheme. The selection of the integration method depends on the stiffness of the ODE. In our case, for example, the Euler integration method was not efficient; however, we could use an integration method that was less complicated than the fourth-order Runge–Kutta method.

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